

# Entropy bounds, monotonicity properties and scaling in CFTs

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## Abstract

We study the ratio of the entropy to the total energy in conformal field theories at finite temperature. For the free field realizations of  $\mathcal{N} = 4$  super Yang-Mills theory in  $D = 4$  and the  $(2, 0)$  tensor multiplet in  $D = 6$ , the ratio is bounded from above. The corresponding bounds are less stringent than the recently proposed Verlinde bound. We show that entropy bounds arise generically in CFTs in connection to monotonicity properties with respect to temperature changes of a generalized  $C$ -function. For strongly coupled CFTs with AdS duals, we show that the ratio obeys the Verlinde bound even in the presence of rotation. For such CFTs, we point out an intriguing resemblance in their thermodynamic formulas with the corresponding ones of two-dimensional CFTs. We show that simple scaling forms for the free energy and entropy of CFTs with AdS duals reproduce the thermodynamical properties of  $(D + 1)$ -dimensional AdS black holes.

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# 1 Introduction

The Bekenstein bound [1] for the ratio of the entropy  $S$  to the total energy  $E$  of a closed physical system<sup>4</sup> that fits in a sphere in three spatial dimensions reads

$$\frac{S}{2\pi RE} \leq 1, \quad (1)$$

where  $R$  denotes the radius of the sphere. Despite many efforts, the microscopic origin of the bound remains elusive. A recent interesting development is Verlinde's observation [3] that CFTs possessing AdS duals satisfy a version of the bound (1). One firstly observes that for general CFTs on  $\mathbb{R} \times S^{D-1}$ , with the radius of  $S^{D-1}$  being  $R$ , the product  $ER$  is independent of the total spatial volume  $V$ . If one defines the sub-extensive part  $E_C$  of the total energy through the scaling property  $E_C(\lambda S, \lambda V) = \lambda^{1-\frac{2}{D-1}} E_C(S, V)$  and  $E = E_{ext} + \frac{1}{2}E_C$ , it follows that<sup>5</sup>

$$E_C = DE - (D-1)TS. \quad (2)$$

The observation of Verlinde is that for strongly coupled CFTs with AdS duals the entropy is given by a generalized Cardy formula

$$S = \frac{2\pi R}{D-1} \sqrt{E_C(2E - E_C)}. \quad (3)$$

To show this, one employs the results for the entropy and total energy of the corresponding  $D$ -dimensional CFT that fits into a  $(D-1)$ -dimensional sphere at finite temperature [4]. These are obtained by virtue of holography [5] from the corresponding thermodynamical quantities of a  $(D+1)$ -dimensional Schwarzschild AdS black hole. From (3) one obtains a bound similar to (1), namely<sup>6</sup>

$$\frac{S}{2\pi RE} \leq \frac{1}{D-1}. \quad (4)$$

In view of the above developments, a natural question arising is whether there exists a microscopic derivation of Verlinde's formula (3) within the thermodynamics of CFTs. This question could

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<sup>4</sup>For recent discussions on general entropy bounds of physical systems see [2] and references therein.

<sup>5</sup>The same result can be obtained using the equation of state  $p = E/V(D-1)$ , where  $p$  is the pressure, that follows from the tracelessness of the energy momentum tensor in a CFT.

<sup>6</sup>Henceforth we call (4) the Verlinde CFT bound to avoid confusion with the cosmological entropy bound suggested by Verlinde also in [3].

be checked in the context of CFTs whose microscopic thermodynamics is well understood, such as free CFTs on  $\mathbb{R} \times S^{D-1}$ . The relevant calculations for dimensions  $D = 4, 6$  were recently undertaken by Kutasov and Larsen [6] (See also [7]. A detailed analysis of four-dimensional thermal CFTs appeared in [8].) They computed the high temperature limits of various partition functions on  $S^1 \times S^{D-1}$ , from which all thermodynamical quantities follow. It was then shown that the Verlinde CFT bound (4) is violated for free CFTs.

In the present work we perform a further analysis of the results in [6] for free CFTs in dimensions  $D = 4, 6$ . We find that for the specific cases of  $\mathcal{N} = 4$   $U(N)$  SYM theory in  $D = 4$  and the  $(2, 0)$  tensor multiplet in  $D = 6$ , the ratio of the entropy to the total energy is bounded from above, however the corresponding bounds are less stringent than (4). We show that general bounds for the ratio of the entropy to the total energy in  $D$ -dimensional CFTs arise naturally under the requirement of monotonicity properties with respect to temperature changes of a generalized  $C$ -function. This generalized  $C$ -function is related to the sub-extensive part (2) of the total energy. Although bounds for the ratio of the entropy to the total energy seem to arise quite generically in CFTs, their exact values depend on the details of the underlying CFT, e.g. it seems that the bounds become more stringent as one goes from weak to strong coupling.

Next we turn our attention to strongly coupled CFTs with AdS duals. We show that the Verlinde formula (3) remains valid also in the case of strongly coupled CFTs in a rotating Einstein universe. We then point out an intriguing resemblance of the formulas of  $(D + 1)$ -dimensional AdS black hole thermodynamics to corresponding formulas in the thermodynamics of two-dimensional CFTs. Particularly interesting is the fact that the entropy of the black hole resembles the  $C$ -function of a two-dimensional system. Motivated by this, we suggest a simple scaling form for the free energy of a  $D$ -dimensional CFT in a space with finite extent at finite temperature. Requiring then that the entropy of such a theory is given by a generalization of the two-dimensional entropy, leads to a simple differential equation whose solution yields a finite-size correlation length that turns out to coincide with the horizon distance of  $(D + 1)$ -dimensional AdS black holes.

The paper is organized as follows. In section 2 we analyze the thermodynamics of free CFTs on  $S^1 \times S^{D-1}$  and show that for  $D = 4$ ,  $\mathcal{N} = 4$  SYM theory and for the  $(2, 0)$  tensor multiplet the ratio of the entropy to the total energy is bounded from above. We also show that bounds

for the ratio of the entropy to the total energy arise naturally in CFTs if certain monotonicity properties of a generalized  $C$ -function are assumed. In section 3 we show that the Verlinde formula (3) is valid for strongly coupled CFTs in a rotating Einstein universe, which are dual to Kerr-AdS black holes. We then suggest simple scaling forms for the free energy and the entropy of  $D$ -dimensional strongly coupled CFTs at finite temperature that reproduce the results of AdS black hole thermodynamics. In section 4 we conclude and discuss some implications of our results for cosmology, as well as possible further developments of our ideas.

## 2 Entropy bounds in CFTs at finite temperature

### 2.1 General results and free CFTs

In this section, we discuss the thermodynamics of conformal field theories. For a general statistical mechanical system, one defines the partition function (we put  $k = \hbar = 1$ )

$$Z = \sum_E \rho(E) e^{-E/T}, \quad (5)$$

where  $\rho(E)$  is the number of states with energy  $E$ . In general, (5) can be evaluated using a saddle point approximation. The exponent is stationary when

$$dS = \frac{dE}{T}, \quad S = \ln \rho, \quad (6)$$

where  $S$  is the entropy of the system. This approximation is valid for a large number of degrees of freedom, i.e. if the underlying theory is a CFT with a large central charge. The free energy

$$F = -T \ln Z, \quad (7)$$

at the saddle point is the exponent in (5),

$$F = E - TS, \quad (8)$$

and on account of (6), the entropy is given by

$$S = -\frac{\partial F}{\partial T}. \quad (9)$$

As an application, consider  $C$  free massless bosons living on a three-dimensional spatial sphere of radius  $R$  at finite temperature. The energy levels and corresponding degeneracies of the various modes are [9, 6]

$$E_n = \frac{n}{R}, \quad d_n = n^2. \quad (10)$$

Therefore, the partition function reads [9, 6]

$$Z_B^{(4)} = \prod_{n=1}^{\infty} (1 - q^n)^{-Cn^2}, \quad q = e^{-2\pi\delta}, \quad \delta = \frac{1}{2\pi RT}. \quad (11)$$

To cast this into the form (5), we exploit the modular properties of the partition function  $Z_B^{(4)}$ . Using a Mellin transform, one obtains [9, 10]

$$Z_B^{(4)} = e^{\frac{\pi C}{360} \delta^{-3}} e^{\frac{\pi C}{120} \delta} \mathcal{Z}, \quad (12)$$

where  $\mathcal{Z}$  is a slowly varying function (approximately constant) near the saddle point. Therefore, the free energy (7) in the saddle point approximation is

$$-F_B^{(4)} R = \frac{C}{240} \left( \frac{1}{3} \delta^{-4} + 1 \right), \quad (13)$$

where we multiplied by the negative radius  $-R$  for convenience. The entropy and energy of the system are easily deduced from the thermodynamical relations (8) and (9),

$$S = \frac{\pi C}{90} \delta^{-3}, \quad E = \frac{C}{240} (\delta^{-4} - 1). \quad (14)$$

For comparison, in two dimensions, the partition function of  $C$  free bosons is

$$Z_B^{(2)} = \prod_{n=1}^{\infty} (1 - q^n)^{-C}, \quad (15)$$

leading to the free energy

$$-F_B^{(2)} R = \frac{C}{24} (\delta^{-2} - 1), \quad (16)$$

while the entropy and energy read, respectively

$$S = \frac{\pi C}{6} \delta^{-1}, \quad ER = \frac{C}{24} (\delta^{-2} + 1). \quad (17)$$

We should point out that in this case there is a contribution from the zero modes of the form  $\delta^{C/2}$ . This is a slowly varying function and does not contribute to the rapidly varying exponentials

that comprise the free energy at the saddle point. The contribution of the zero modes becomes significant when the saddle-point approximation breaks down, for a small central charge.<sup>7</sup> Here, we are interested in the large  $C$  limit, so such contributions will be ignored.

Eq. (17) implies the Cardy formula [11]

$$S = 2\pi \sqrt{\frac{C}{6} \left( E - \frac{C}{24} \right)}, \quad (18)$$

and the Bekenstein bound (1) for the ratio

$$\frac{S}{2\pi ER} = \frac{2\delta}{1 + \delta^2} \leq 1. \quad (19)$$

The above results also hold for fermions, because the free energy for a fermion is  $F_F^{(2)} = \frac{1}{2} F_B^{(2)}$ . Returning to four dimensions, we note from (14) that there is a transition point at  $\delta = 1$  where  $ER = 0$ . At that point the ratio  $S/E$  diverges because the entropy remains finite. Thus, it seems as if the ratio of the entropy to the total energy is not bounded for free bosons. Similar results hold for Weyl fermions and vector bosons. However, in systems with diverse mode species there is a chance that the above ratio is bounded. Consider the free energy of a system of  $N_B$  bosons,  $N_F$  Weyl fermions and  $N_V$  vectors that reads [6]

$$-FR = a_4 \delta^{-4} + a_2 \delta^{-2} + a_0, \quad (20)$$

where

$$a_4 = \frac{1}{720}(N_B + \frac{7}{4} N_F + 2N_V), \quad a_2 = -\frac{1}{24}(\frac{1}{4} N_F + 2N_V), \quad a_0 = \frac{1}{240}(N_B + \frac{17}{4} N_F + 22N_V), \quad (21)$$

satisfying the constraint  $3a_4 = a_2 + a_0$ . The entropy and energy are, respectively,

$$S = 2\pi (4a_4 \delta^{-3} + 2a_2 \delta^{-1}), \quad ER = 3a_4 \delta^{-4} + a_2 \delta^{-2} - a_0. \quad (22)$$

The Bekenstein-Verlinde ratio is

$$\frac{S}{2\pi ER} = \delta \frac{4a_4 + 2a_2 \delta^2}{3a_4 + a_2 \delta^2 - a_0 \delta^4} = \frac{2\delta}{1 + \delta^2} \frac{2a_4 + a_2 \delta^2}{3a_4 - a_0 \delta^2}. \quad (23)$$

Remarkably, for the  $\mathcal{N} = 4$  SYM model, we have  $a_2 = -6a_4$ , which implies

$$\frac{S}{2\pi ER} = \frac{2}{3} \frac{2\delta}{1 + \delta^2}. \quad (24)$$

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<sup>7</sup>We thank D. Kutasov for pointing out this to us.

One might now think that (24) generally implies the bound  $S/(2\pi ER) \leq 2/3$ . However, we have to keep in mind that (20) and (22) are high temperature expansions and should not be trusted for large- $\delta$ . Starting from high temperatures (small- $\delta$ ), there is a critical point at which both  $S$  and  $ER$  vanish for

$$\delta_c^2 = -\frac{2a_4}{a_2} = \frac{1}{3}. \quad (25)$$

We should not expect that (23) makes sense for  $\delta \geq \delta_c$ . Nevertheless, for  $\delta \leq \delta_c$ , we obtain the bound

$$\frac{S}{2\pi ER} \leq \frac{\sqrt{3}}{3}, \quad (26)$$

which is weaker than the Verlinde CFT bound (4)  $S/(2\pi ER) \leq 1/3$ .

It is perhaps worth mentioning that if one imposes periodic boundary conditions on the gaugino, as suggested by Tseytlin [12] to account for the disagreement on the number of degrees of freedom between the weak and strong coupling regimes, the above results still hold. Indeed, the partition function for a gaugino with periodic boundary conditions is

$$\tilde{Z}_F = \prod_{n=0}^{\infty} (1 - q^{n+1/2})^{2n(n+1)}, \quad (27)$$

which leads to the free energy for  $\tilde{N}_F$  gauginos,

$$\tilde{F}R = \tilde{N}_F(\tilde{a}_4\delta^{-4} + \tilde{a}_2\delta^{-2} + \tilde{a}_0), \quad (28)$$

where

$$\tilde{a}_4 = -\frac{1}{360}, \quad \tilde{a}_2 = \frac{1}{48}, \quad \tilde{a}_0 = -\frac{7}{240}. \quad (29)$$

For the  $\mathcal{N} = 4$  SYM model with such gauginos, the coefficients of the free energy (20) still satisfy  $3a'_4 = a'_2 + a'_0$  and the ratio  $a'_2/a'_4$  is unchanged

$$\frac{a'_2}{a'_4} = -30 \frac{\frac{1}{4} N_F - \frac{1}{2} \tilde{N}_F + 2N_V}{N_B + \frac{7}{4} N_F - 2\tilde{N}_F + 2N_V} = -6. \quad (30)$$

Thus, the only effect of imposing periodic boundary conditions on the gauginos is the reduction of the free energy by an overall factor of  $a'_4/a_4 = 3/4$ .

Next we turn to the  $(2,0)$  tensor multiplet in  $D = 6$ . The free energy is [6]

$$-FR = a_6\delta^{-6} + a_4\delta^{-4} + a_2\delta^{-2} + a_0. \quad (31)$$

The entropy is given by

$$\frac{S}{2\pi} = T S R \delta = -T \frac{\partial(FR)}{\partial T} \delta = 6a_6 \delta^{-5} + 4a_4 \delta^{-3} + 2a_2 \delta^{-1}, \quad (32)$$

and the energy by

$$ER = 5a_6 \delta^{-6} + 3a_4 \delta^{-4} + a_2 \delta^{-2} - a_0. \quad (33)$$

Considering the same ratio as before we obtain

$$\frac{S}{2\pi RE} = \delta \frac{6a_6 + 4a_4 \delta^2 + 2a_2 \delta^4}{5a_6 + 3a_4 \delta^2 + a_2 \delta^4 - a_0 \delta^6}. \quad (34)$$

The coefficients are related through [6]

$$5a_6 - 3a_4 + a_2 + a_0 = 0, \quad (35)$$

and we have  $a_6, a_2 > 0$ ,  $a_4, a_0 < 0$ . The denominator can be factorized into

$$5a_6 + 3a_4 \delta^2 + a_2 \delta^4 - a_0 \delta^6 = (1 + \delta^2)(5a_6 + (a_2 + a_0)\delta^2 - a_0 \delta^4). \quad (36)$$

Using the explicit values  $a_6 = 1/5$ ,  $a_4 = -5/3$ ,  $a_2 = 19$ ,  $a_0 = -25$  we obtain

$$\frac{S}{2\pi ER} = \frac{2\delta}{1 + \delta^2} \frac{\frac{3}{5} - \frac{10}{3}\delta^2 + 19\delta^4}{1 - 6\delta^2 + 25\delta^4}. \quad (37)$$

This is a well-behaved function of  $\delta$ , which has a maximum of 0.824. We therefore conclude that

$$\frac{S}{2\pi ER} \leq 0.824, \quad (38)$$

which is less stringent than the Verlinde CFT bound (4).

## 2.2 General entropy bounds in CFTs from monotonicity properties

In this subsection we show that entropy bounds in CFTs at finite temperature arise naturally from the monotonicity property of a generalized  $C$ -function [14]. On general grounds, the free energy of a  $D$ -dimensional statistical system that can be described in its continuum limit by a renormalized field theory [15] can be written as

$$F(T, V, g_R) = E_0(V, g_R) - T^D \mathcal{C}(T, V, g_R), \quad (39)$$



where  $V$  is the total volume,  $g_R$  denote collectively a set of renormalized couplings and  $E_0(V, g_R)$  is the zero temperature energy of the system. For two-dimensional systems, the function  $\mathcal{C}(T, V, g_R)$  is proportional to Zamolodchikov's  $C$ -function [16] and for unitary quantum field theories it is a monotonically decreasing function along the RG-flow from the UV to the IR. For systems at finite temperature, however, one might also consider a change in the temperature at some fixed values of the coupling constants. Then, the question arising is how the above generalized  $C$ -function behaves in such a case. In a temperature interval where no phase transitions occur, a natural assumption is that the generalized  $C$ -function above behaves monotonically under temperature changes. For example, the  $C$ -function defined in (39) is proportional to the quantity used in [17] as a measure of the massless degrees of freedom coupled at a fixed point. In that case the IR and UV fixed points were taken, respectively, to be the  $T \rightarrow 0$  and  $T \rightarrow \infty$  limits of (39). One then expects that the  $C$ -function above describes the process of thermal excitation of more and more degrees of freedom as the temperature is raised, in which case it seems natural to assume that

$$T \frac{\partial}{\partial T} \mathcal{C}(T) \geq 0. \quad (40)$$

Such a simple picture is consistent with the fact that the free energy density is minus the pressure. The monotonicity property (40) leads to a general bound for the ratio of the entropy to the total energy of the above statistical system. From (39) we obtain after some simple algebra

$$S = T^{D-1} \left[ D\mathcal{C} + T \frac{\partial \mathcal{C}}{\partial T} \right], \quad (41)$$

$$E - E_0 = T^D \left[ (D-1)\mathcal{C} + T \frac{\partial \mathcal{C}}{\partial T} \right], \quad (42)$$

$$T \frac{\partial \mathcal{C}}{\partial T} = T^{-D} [D(E - E_0) - (D-1)TS]. \quad (43)$$

From (41) and (42) we see that our definition of  $\mathcal{C}$  is consistent with the third law of thermodynamics which requires that  $\lim_{T \rightarrow 0} S = 0$ . We then easily see from (43) that the monotonicity property (40) leads to the bound

$$\frac{S}{2\pi R(E - E_0)} \leq \frac{D}{D-1} \delta, \quad (44)$$

with the same  $\delta = (2\pi RT)^{-1}$  as in the previous subsection. As it was discussed in [14], the bound (44) implies that the number of states with energy between  $E(T)$  and  $E(T + \Delta T)$  in the

underlying system is bounded from above. Such a property does not follow from the basic laws of thermodynamics.

The bound (44) also makes no reference to the specific properties of the underlying thermal CFT. For additional information one has to deal directly with a particular CFT model, such as the free field theories of subsection 2.1 or the strongly coupled CFTs of subsection 3.1. Nevertheless, comparing (2) and (43) we see that for systems with zero ground state energy (such as supersymmetric systems, cf. e. g. [6]), one has

$$T \frac{\partial \mathcal{C}}{\partial T} = T^{-D} E_C. \quad (45)$$

Formula (45) relates the derivative of the generalized  $C$ -function to the sub-extensive part of the total energy and gives useful physical insight for the latter. For example, the property  $E_C \geq 0$  which was required in [6] to give meaning to the Verlinde formula (3) is now equivalent to (40). Furthermore, the fact that  $E_C < 0$  for the massless free boson in  $D = 4$  [6] can be attributed, by virtue of (45), to the temperature instability of the vacuum of that particular theory in which case we should not expect the analysis leading to formula (3) to be valid.

## 3 Entropy bounds in CFTs with AdS duals

### 3.1 General results

In this section we turn our attention to strongly coupled CFTs in  $D$ -dimensions, possessing AdS duals. The thermodynamics of such theories follows quite generally from the thermodynamics of  $(D + 1)$ -dimensional AdS black holes, through holography. We consider the rotating Kerr-AdS (KAdS) black hole in  $(D + 1)$ -dimensions given by <sup>8</sup> [18]

$$ds^2 = -\frac{\Delta_r}{\rho^2} [dt - \frac{a}{\Xi} \sin^2 \theta d\phi]^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} [adt - \frac{r^2 + a^2}{\Xi} d\phi]^2 + r^2 \cos^2 \theta d\Omega_{D-3}^2, \quad (46)$$

where  $d\Omega_{D-3}^2$  denotes the standard metric on the unit  $S^{D-3}$  and

$$\Delta_r = (r^2 + a^2) \left( 1 + \frac{r^2}{R^2} \right) - 2Mr^{4-D},$$

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<sup>8</sup>For simplicity we restrict ourselves to the case of only one rotation parameter.

$$\Delta_\theta = 1 - \frac{a^2}{R^2} \cos^2 \theta, \quad (47)$$

$$\Xi = 1 - \frac{a^2}{R^2},$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta. \quad (48)$$

The inverse temperature, free energy, entropy, energy and angular momentum read [18, 19]

$$\beta = \frac{4\pi(r_+^2 + a^2)}{(D-2) \left(1 + \frac{a^2}{R^2}\right) r_+ + \frac{Dr_+^3}{R^2} + \frac{(D-4)a^2}{r_+}}, \quad (49)$$

$$F = -\frac{V_{D-1}}{16\pi G_{D+1}\Xi} r_+^{D-4} (r_+^2 + a^2) \left( \frac{r_+^2}{R^2} - 1 \right), \quad (50)$$

$$S = \frac{V_{D-1}}{4G_{D+1}\Xi} r_+^{D-3} (r_+^2 + a^2), \quad (51)$$

$$E = \frac{(D-1)V_{D-1}}{16\pi G_{D+1}\Xi} r_+^{D-4} (r_+^2 + a^2) \left( \frac{r_+^2}{R^2} + 1 \right), \quad (52)$$

$$J = \frac{aV_{D-1}}{8\pi G_{D+1}\Xi^2} r_+^{D-4} (r_+^2 + a^2) \left( \frac{r_+^2}{R^2} + 1 \right), \quad (53)$$

where  $G_{D+1}$  is Newton's constant,  $V_{D-1}$  denotes the volume of the unit  $S^{D-1}$ ,  $r_+$  (the horizon radial coordinate) is the largest root of  $\Delta_r = 0$ , and the rotation parameter  $a$  is restricted to the range  $0 \leq a < R$ . According to the AdS/CFT duality conjecture [4], the above thermodynamical quantities are associated to a strongly coupled  $D$ -dimensional CFT residing on the conformal boundary of the spacetime (46), i. e. on a rotating Einstein universe.

Defining  $\Delta = R/r_+$ , and the Bekenstein entropy  $S_B = 2\pi ER/(D-1)$ , we obtain from (51) and (52)

$$\frac{S}{S_B} = \frac{2\Delta}{1 + \Delta^2} \leq 1. \quad (54)$$

The Bekenstein bound is saturated for  $\Delta = 1$ , i. e. at the Hawking-Page transition point [13], where the free energy becomes zero. We further note that we can write

$$2ER = \frac{D-1}{2\pi} S \frac{R}{r_+} [\Delta^{-2} + 1], \quad (55)$$

which, for arbitrary  $D$ , is exactly the behavior of a two-dimensional CFT (17) with characteristic scale  $R$ , temperature  $\tilde{T} = 1/(2\pi R\Delta) = r_+/(2\pi R^2)$ , and central charge proportional to  $SR/r_+$ . This resemblance motivates us to define the Casimir energy as the sub-extensive part of (55), i. e.

$$E_C = \frac{(D-1)}{2\pi} \frac{S}{r_+} = \frac{D-1}{2\pi} \frac{V_{D-1}}{4G_{D+1}\Xi} r_+^{D-4} (r_+^2 + a^2). \quad (56)$$

Note that in the non-rotating case  $a = 0$ , (56) coincides with the expression <sup>9</sup> given in [3].

One now easily verifies that the quantities (51), (52) and (56) satisfy exactly the Verlinde formula (3). We can also define the "Casimir entropy" [3] by

$$S_C = \frac{2\pi}{D-1} E_C R = S \frac{R}{r_+}, \quad (57)$$

which allows to write the free energy as

$$-FR = \frac{S_C}{4\pi} [\Delta^{-2} - 1]. \quad (58)$$

Comparing (58) with the corresponding relation of a two-dimensional bosonic CFT (16), we see that the Casimir entropy  $S_C$  is essentially proportional to the central charge [3], or equivalently to the number of degrees of freedom coupled at the critical point.

Within such an interpretation for  $S_C$  we can now see that the temperature  $\tilde{T} = r_+/(2\pi R^2)$  makes thermodynamic sense as a temperature of a two-dimensional system. Considering for simplicity the case when  $a = 0$  and constant volume, the second law of black hole mechanics reads

$$dE(S, N) = TdS + \mu dN, \quad (59)$$

where by virtue of (57) and (51) we defined the number of degrees of freedom (generalized central charge) as

$$N = \frac{V_{D-1} R^{D-1}}{16\pi G_{D+1}} = \frac{S_C}{4\pi} \left( \frac{R}{r_+} \right)^{D-2}, \quad (60)$$

and  $\mu$  is the chemical potential. After some algebra, we find

$$\mu = - \left( \frac{r_+}{R} \right)^{D-2} \left( \frac{r_+^2}{R^2} - 1 \right) \frac{1}{R}. \quad (61)$$

The free energy (58) can then be written simply as

$$F = \mu N. \quad (62)$$

Furthermore, for  $S_C = \text{const.}$ , we have

$$d\tilde{E} = \tilde{T} dS, \quad (63)$$

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<sup>9</sup>The expressions in [3] are related to ours by  $L = R^2/r_+$  and  $\frac{c}{12} \frac{V}{R^{2D-2}} = \frac{V_{D-1}}{4G_{D+1}}$ .

where

$$\tilde{E} = E \frac{1}{D-1} \quad , \quad \tilde{T} = \frac{1}{D-1} \left( T + \mu \frac{dN}{dS} \right) = \frac{r_+}{2\pi R^2}. \quad (64)$$

Notice that  $\tilde{E} = E$  and  $\tilde{T} = T$  for  $D = 2$ , as expected.

Remarkably enough, these results generalize to the  $a \neq 0$  case. Even with the addition of one more potential and the attendant generalization of the second law of black hole mechanics to

$$dE = TdS + \mu dN + \Omega dJ, \quad (65)$$

the condition  $dS_C = 0$  still describes a two-dimensional system.

Finally, as in [3] we define the "Bekenstein-Hawking energy"  $E_{BH}$  as the energy for which the black hole entropy  $S$  (51) and the Bekenstein entropy  $S_B$  are equal,  $2\pi E_{BH} R / (D-1) = S$ , yielding

$$E_{BH} = E_C \frac{r_+}{R}. \quad (66)$$

One checks that  $E_{BH} \leq E$ . Furthermore, because we are above the Hawking-Page transition point, we have  $r_+ \geq R$ , and therefore

$$E_C \leq E_{BH} \leq E, \quad S_C \leq S \leq S_B, \quad (67)$$

where equality holds when the HP phase transition is reached. As the entropy  $S$  is a monotonically increasing function of  $E_C$  (or, equivalently, of  $r_+$ ), the maximum entropy is reached when  $E_C = E_{BH}$ , i. e. at the HP phase transition  $r_+ = R$ . It is quite interesting to observe that at this point the central charge  $c/12 = S_C/(2\pi)$  takes e. g. for  $D = 4$ ,  $\mathcal{N} = 4$   $U(N)$  SYM theory the value<sup>10</sup>  $c = 6N^2$ . This is exactly the central charge of a two-dimensional free CFT containing the  $6N^2$  scalars of  $D = 4$ ,  $\mathcal{N} = 4$  SYM.

### 3.2 Scaling form for the free energy and entropy of CFTs with AdS duals

The results of the previous subsection suggest a simple scaling form for the free energy of strongly coupled  $D$ -dimensional CFTs at finite temperature. One motivation comes from expression (49),

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<sup>10</sup>Here we used the AdS/CFT dictionary  $N^2 = \frac{\pi R^3}{2G_5}$ .

for  $a = 0$ , which takes the form<sup>11</sup>

$$r_+(T, R) = R \frac{2\pi RT}{D} \left[ 1 + \sqrt{1 - \frac{D(D-2)}{(2\pi RT)^2}} \right], \quad (68)$$

that resembles the finite-size scaling of the correlation length - here  $r_+(T, R)$  - in a system with finite size  $R$  at temperature  $T$  [15, 20]. For such a system, the relation  $r_+(T, R) = R$  defines the *rounding temperature* [20] which is an approximation to the true critical temperature. From the above, we see that the Hawking-Page temperature  $T_{HP} = (D-1)/2\pi R$  [4] coincides with the rounding temperature of the finite-size system.

Interpreting  $r_+(T, R)$  as the correlation length of a system with finite size  $R$  at temperature  $T$  makes all thermodynamical relations derived in the previous subsection similar to finite-size scaling. In particular, the basic assumption of finite-size scaling (see e.g. [20, 21]), that there exists only one length scale in the theory is consistent with the fact that the dimensionless quantities  $ER$ ,  $S$  and  $FR$  in (55), (51) and (58) are all given in terms of the ratio  $\Delta = R/r_+$ . In this sense, one could have started by postulating certain simple scaling relations for the above thermodynamical quantities which would then describe an underlying system of finite extent  $R$  at temperature  $T$ . As an example, in  $D$  dimensions we could have postulated the following scaling relation for the dimensionless quantity  $FR$ , in the regime  $r_+ > R$ ,

$$FR = G^{(2)}(\Delta)\Delta^{2-D}, \quad (69)$$

where the function  $G^{(2)}(x)$  is given by

$$G^{(2)}(x) = K(1 - x^{-2}), \quad (70)$$

and  $24K$  is a constant playing the role of a *generalized central charge*. Clearly, (69) is a simple generalization of the two-dimensional relation (16). Such a relation implies that in the finite-size scaling regime where  $R, r_+ \rightarrow \infty$  but  $\Delta$  finite, the free energy density  $f$  (free energy per unit spatial volume) behaves as  $f \sim R^{-D}$ , as it should for a  $D$ -dimensional system with finite size  $R$ .

Next we have to make sure that (69) satisfies the basic thermodynamical equation at constant volume

$$\frac{\partial}{\partial T} F = \frac{\partial \Delta}{\partial T} \frac{\partial}{\partial \Delta} F = -S. \quad (71)$$

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<sup>11</sup>We keep only the positive root, the negative one being related to a branch of unstable black holes.

Requiring then a simple scaling form for  $S$  would give a differential equation that could determine  $\Delta$  and consequently  $r_+(T, R)$ . We can check this in  $D = 2$  where  $K$  in (69) is proportional to the central charge of the theory. In this case we know that (see e.g. (17))

$$S = S^{(2)} \equiv 4\pi K \Delta^{-1}. \quad (72)$$

Plugging this into (71) we easily obtain  $\Delta = (2\pi RT)^{-1}$  which is the standard two-dimensional result.

The two-dimensional relation (72) has a nice physical content away from the critical point as it relates the two-dimensional  $C$ -function, which corresponds to the off-critical value of  $24K$ , to the entropy of the system. Had we wanted to keep such a physical picture in higher dimensions, we should require a form for the entropy similar to (72). To this end we suggest that in the  $D$ -dimensional system the following generalization of the two-dimensional result (72) should hold

$$S = S^{(2)} \Delta^{2-D}. \quad (73)$$

Plugging this into (71) we obtain

$$\left[ (D-2) - D \frac{1}{\Delta^2} \right] \frac{\partial \Delta}{\partial T} = 4\pi R, \quad (74)$$

whose solution finally yields

$$Dr_+^2 - 4\pi R^2 T r_+ + R^2(D-2) = 0. \quad (75)$$

The above is exactly the relation (49) (for  $a = 0$ ), coming from the study of AdS black hole thermodynamics.

## 4 Conclusions and Discussion

In the present work we studied the entropy bounds in  $D$ -dimensional CFTs both at the free field theory level as well as at strong coupling. Using the results of [6] we showed that the ratio of the entropy to the total energy is bounded for the free field realizations of  $\mathcal{N} = 4$   $U(N)$  SYM theory in  $D = 4$  and the  $(2, 0)$  tensor multiplet in  $D = 6$ . We pointed out that general bounds for the entropy to energy ratio in CFTs at finite temperature follow from the requirement of

monotonicity of a generalized  $C$ -function with respect to temperature changes. We showed that such a generalized  $C$ -function is related to the sub-extensive part of the total energy. Then we showed that for CFTs in a rotating Einstein universe possessing AdS duals, the Verlinde entropy formula (3) is still valid. We further suggested that if we interpret the horizon distance  $r_+(T, R)$  as a correlation length, formulas (49)-(53) (for  $a = 0$ ) describe the thermodynamics of a  $D$ -dimensional statistical system of finite extent  $R$  at finite temperature. The rounding temperature, which is an approximation to the critical temperature, of such a system is given by the Hawking-Page transition temperature. Assuming then simple scaling forms for the free energy and the entropy of the system, yields an explicit formula for the correlation length  $r_+(T, R)$  which coincides with the result (49) coming from the thermodynamics of AdS black holes.

Let us briefly mention one implication of our results for cosmology. As in in Ref. [3], consider a radiation dominated closed Friedman-Robertson-Walker (FRW) universe. The FRW metric takes the form

$$ds^2 = -d\tau^2 + \mathcal{R}^2(\tau)d\Omega_{D-1}^2, \quad (76)$$

where  $\mathcal{R}(\tau)$  represents the radius of the universe at a given time  $\tau$ . Note that the metric (76) is conformally equivalent to

$$d\tilde{s}^2 = -dt^2 + R^2 d\Omega_{D-1}^2, \quad (77)$$

where  $dt = R d\tau/\mathcal{R}(\tau)$ . If the radiation is described by a CFT, one can equally well use (77) instead of (76). If, in addition, this CFT admits an AdS dual, it can be described by a Schwarzschild-AdS black hole at some temperature  $T$ , because (77) is precisely the metric on the conformal boundary of (46) for  $a = 0$ <sup>12</sup>. The observations made by Verlinde [3] concerning entropy, energy and temperature bounds in a radiation dominated universe then fit nicely into this AdS black hole description. In particular, the universe is weakly (strongly) self-gravitating if  $H\mathcal{R} \leq 1$  ( $H\mathcal{R} \geq 1$ ), where  $H = \dot{\mathcal{R}}/\mathcal{R}$  denotes the Hubble constant, and the dot refers to differentiation with respect to  $\tau$ . One has  $H\mathcal{R} = 1$  iff the Bekenstein-Hawking entropy  $S$  equals the Bekenstein entropy  $S_B$ . We saw above that this happens precisely at the HP transition point  $r_+ = R$ , so the borderline between the weakly and strongly self-gravitating regime is the Hawking-Page phase transition temperature  $T_{HP} = (D-1)/2\pi R$ . This identification makes indeed sense, because

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<sup>12</sup>In what follows, we shall only consider the static case  $a = 0$ .



below  $T_{HP}$  (weakly gravitating) one has AdS space filled with thermal radiation which collapses above  $T_{HP}$  ( $r_+ \geq R$ , strongly gravitating) to form a black hole. Furthermore, in [3] a limiting temperature was found for the early universe,

$$T \geq T_H = -\frac{\dot{H}}{2\pi H} \quad \text{for } H\mathcal{R} \geq 1. \quad (78)$$

We conclude therefore that Verlinde's limiting temperature  $T_H$  corresponds to the temperature  $T_{HP}$  where the HP phase transition takes place.

Concerning further developments of our ideas, it might be interesting to study our *generalized central charge* defined in (60). We discussed in the text that this quantity intriguingly resembles a standard two-dimensional central charge. Such an interpretation leads to the conjecture that there might exist a two-dimensional CFT model whose dynamics in the presence of irrelevant operators, as follows from (69) and (73), underlies the dynamics of the  $D$ -dimensional CFTs possessing AdS duals. Such a conjecture might explain the fact that the latter theories share unexpectedly many of the properties of two-dimensional CFTs [22].

It would also be interesting to understand what happens if the sub-extensive part of the total energy becomes negative. This is e. g. the case for hyperbolic AdS black holes. As  $E_C$  corresponds somehow to a central charge, this would indicate that the underlying CFT is non-unitary or that the theory has a thermally unstable ground state.

Another point that needs to be understood is the underlying model which produces the simple scaling relations for the free energy and the entropy described in section 3.2. This might be of some interest as the scaling relations (69) and (73) seem to have a wide application range. Different choices of the scaling function (70) would lead to formulas for the correlation length  $r_+(R, T)$  different from (75). As an example, one could try to generalize relations (13) and (14) for the free energy and entropy of the massless free boson in  $D = 4$ . A possible generalization in the spirit of (69) and (73) would give the free energy and entropy of a  $D$ -dimensional CFT as

$$FR = \mathcal{K}(1 + \frac{1}{3}\Delta^{-4})\Delta^{4-D}, \quad S = \frac{8\pi\mathcal{K}}{3}\Delta^{1-D}. \quad (79)$$

Then, one can show that imposition of the basic thermodynamical relation (71) leads to a differential equation whose solution yields

$$Dr_+^4 - 24\pi R^2 T r_+^3 - (D - 4)R^4 = 0. \quad (80)$$

It remains to be seen if  $r_+$  in (80) corresponds to the horizon distance of some new kind of black holes in dimensions  $D + 1 \geq 5$ .

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